Matrices & Determinant

As for everything else, so for a mathematical theory, beauty can be perceived but not explained..... Cayley Arthur

Introduction :

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix**. If a matrix has m rows and n columns then the **order** of matrix is written as m × n and we call it as order m by n

The general m × n matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1j} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}$$

where a_{ij} denote the element of i^{th} row & j^{th} column. The above matrix is usually denoted as $[a_{ij}]_{m \times n}$.

Notes :

(i) The elements a_{11}, a_{22}, a_{33},........ are called as **diagonal elements**. Their sum is called as **trace of A** denoted as tr(A)

(ii) Capital letters of English alphabets are used to denote matrices.

(iii) Order of a matrix : If a matrix has m rows and n columns, then we say that its order is "m by n", written as "m × n".

Example # 1 : Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution : In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

$a_{ij} = \frac{1}{2}|i - 3j|$, i = 1, 2, 3 and j = 1, 2

Therefore $a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1$ $\quad a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$

$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2}$ $\quad a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$

$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0$ $\quad a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$

Hence the required matrix is given hy $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

Types of Matrices :

Row matrix :

A matrix having only one row is called as row matrix (or row vector).General form of row matrix is $A = [a_{11}, a_{12}, a_{13}, \ldots, a_{1n}]$

This is a matrix of order "1 × n" (or a row matrix of order n)

Column matrix :

A matrix having only one column is called as column matrix (or column vector).

Column matrix is in the form $A = \begin{bmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{bmatrix}$

This is a matrix of order "m × 1" (or a column matrix of order m)

Zero matrix :

$A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0 \ \forall$ i & j.

e.g. : (i) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Square matrix :

A matrix in which number of rows & columns are equal is called a square matrix. The general form of a square matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$$ which we denote as $A = [a_{ij}]_n$.

This is a matrix of order "n × n" (or a square matrix of order n)

Diagonal matrix :

A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e., all the elements of the square matrix other than diagonal elements are zero)

Note : Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, ...a_{nn})$.

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$

Scalar matrix :

Scalar matrix is a diagonal matrix in which all the diagonal elements are same. $A = [a_{ij}]_n$ is a scalar matrix, if (i) $a_{ij} = 0$ for $i \neq j$ and (ii) $a_{ij} = k$ for $i = j$.

e.g. : (i) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

Unit matrix (identity matrix) :

Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I).

i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ii} = 1$

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Upper triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be upper triangular, if $a_{ij} = 0$ for $i > j$ (i.e., all the elements below the diagonal elements are zero).

e.g. : (i) $\begin{bmatrix} a & b & c & d \\ 0 & x & y & z \\ 0 & 0 & u & v \end{bmatrix}$ (ii) $\begin{bmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix}$

Lower triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for $i < j$. (i.e., all the elements above the diagonal elements are zero.)

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & y & z \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ x & y & z & 0 \end{bmatrix}$

Comparable matrices : Two matrices A & B are said to be comparable, if they have the same order (i.e., number of rows of A & B are same and also the number of columns).

e.g. : (i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ are comparable

e.g. : (ii) $C = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $D = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ are not comparable

Equality of matrices :

Two matrices A and B are said to be equal if they are comparable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$

A = B iff (i) $m = p$, $n = q$

(ii) $a_{ij} = b_{ij}$ $\forall$ i & j.

Example # 2 : Let $A = \begin{bmatrix} \sin\theta & 1/\sqrt{2} \\ -1/\sqrt{2} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ & $B = \begin{bmatrix} 1/\sqrt{2} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$. Find θ so that A = B.

Solution : By definition A & B are equal if they have the same order and all the corresponding elements are equal.

Thus we have $\sin\theta = \dfrac{1}{\sqrt{2}}$, $\cos\theta = -\dfrac{1}{\sqrt{2}}$ & $\tan\theta = -1$

$\Rightarrow \quad \theta = (2n + 1)\pi - \dfrac{\pi}{4}$.

Example # 3 : If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$, then find the values of a, b, c, x, y and z.

Solution : As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$x + 3 = 0$ $\quad z + 4 = 6$ $\quad 2y - 7 = 3y - 2$

$a - 1 = -3$ $\quad 0 = 2c + 2$ $\quad b - 3 = 2b + 4$

$\Rightarrow$ $a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$

Multiplication of matrix by scalar :

Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$ & $-3A \equiv (-3)A = \begin{bmatrix} -6 & 3 & -9 & -15 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$

Note : If A is a scalar matrix, then $A = \lambda I$, where λ is a diagonal entry of A

Addition of matrices :

Let A and B be two matrices of same order (i.e. comparable matrices). Then A + B is defined to be.

$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$.
$= [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -2 & -3 \\ 5 & 7 \end{bmatrix}$, $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 6 & 7 \end{bmatrix}$

Substraction of matrices :

Let A & B be two matrices of same order. Then A – B is defined as A + (– B) where – B is (– 1) B.

Properties of addition & scalar multiplication :

Consider all matrices of order m × n, whose elements are from a set F (F denote Q, R or C).
Let $M_{m \times n}$ (F) denote the set of all such matrices.
Then

(a) $A \in M_{m \times n}(F)$ & $B \in M_{m \times n}(F)$ $\Rightarrow$ $A + B \in M_{m \times n}(F)$
(b) $A + B = B + A$
(c) $(A + B) + C = A + (B + C)$
(d) $O = [o]_{m \times n}$ is the additive identity.
(e) For every $A \in M_{m \times n}(F)$, $-A$ is the additive inverse.
(f) $\lambda(A + B) = \lambda A + \lambda B$
(g) $\lambda A = A\lambda$
(h) $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$

Example # 4 : IF $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B

Solution : We have 2A + 3X = 5B.

$\Rightarrow$ $3X = 5B - 2A$

$\Rightarrow$ $X = \frac{1}{3}(5B - 2A)$

$$\Rightarrow \quad X = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$\Rightarrow \quad X = \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -13 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Multiplication of matrices :

Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$, which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 3 & 4 & 9 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

Notes : (1) The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined $\not\Rightarrow$ BA is defined.

(2) In general AB ≠ BA, even when both the products are defined.

(3) A (BC) = (AB) C, whenever it is defined.

Properties of matrix multiplication :

Consider all square matrices of order 'n'. Let M_n (F) denote the set of all square matrices of order n. (where F is Q, R or C). Then

(a) $A, B \in M_n (F) \Rightarrow AB \in M_n (F)$

(b) In general AB ≠ BA

(c) (AB) C = A(BC)

(d) I_n, the identity matrix of order n, is the multiplicative identity.
$AI_n = A = I_n A \quad \forall A \in M_n (F)$

(e) For every non singular matrix A (i.e., |A| ≠ 0) of M_n (F) there exist a unique (particular) matrix $B \in M_n$ (F) so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar (λA) B = λ(AB) = A(λB).

(g) $A(B + C) = AB + AC \quad \forall A, B, C \in M_n (F)$

(h) $(A + B) C = AC + BC \quad \forall A, B, C \in M_n (F)$.

Notes : (1) Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

(2) For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Example # 5 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40 I = O$

Solution : We have $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

So $A^3 = AA^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$

Now $A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 90-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Self practice problems :

(1) If $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, verify that $A(\alpha)\,A(\beta) = A(\alpha + \beta)$.
Hence show that in this case $A(\alpha).\,A(\beta) = A(\beta)\,.\,A(\alpha)$.

(2) Let $A = \begin{bmatrix} 4 & 6 & -1 \\ 3 & 0 & 2 \\ 1 & -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$ and $C = [3\ \ 1\ \ 2]$.
Then which of the products ABC, ACB, BAC, BCA, CAB, CBA are defined. Calculate the product whichever is defined.
Answer (2) Only CAB is defined. CAB = [25 100]

Transpose of a matrix :

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A′(or A^T) and is defined as
$A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji}$ $\forall$ i & j.
i.e. A′ is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ x & y & z & w \end{bmatrix}$, $A' = \begin{bmatrix} 1 & a & x \\ 2 & b & y \\ 3 & c & z \\ 4 & d & w \end{bmatrix}$

Results :

(i) For any matrix $A = [a_{ij}]_{m \times n}$, $(A')' = A$

(ii) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(iv) $(A_1 \pm A_2 \pm ..... \pm A_n)' = A_1' \pm A_2' \pm ..... \pm A_n'$, where A_i are comparable.

(v) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(vi) $(A_1\, A_2\, ...A_n)' = A_n'.\, A_{n-1}'\, ...A_2'\, .\, A_1'$, provided the product is defined.

Symmetric & skew-symmetric matrix :

A square matrix A is said to be symmetric if A′ = A
i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}$ $\forall$ i & j.

A square matrix A is said to be skew-symmetric if A′ = – A
i.e. Let $A = [a_{ij}]_n$. A is skew-symmetric iff $a_{ij} = -a_{ji}$ $\forall$ i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Notes : (1) In a skew-symmetric matrix all the diagonal elements are zero.
($\because$ $a_{ii} = -a_{ii}$ $\Rightarrow$ $a_{ii} = 0$)

(2) For any square matrix A, A + A′ is symmetric & A – A′ is skew-symmetric.

(3) Every square matrix can be uniqualy expressed as a sum of two square matrices of which one is symmetric and the other is skew-symmetric.

A = B + C, where $B = \frac{1}{2}(A + A')$ & $C = \frac{1}{2}(A - A')$.

Example # 6 : If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, B = [1 3 – 6], verify that (AB)' = B'A'.

Solution : We have

$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, B = [1 3 –6]

Then $AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \;\; 3 \;\; -6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

Now A' = [–2 4 5], $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \;\; 4 \;\; 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$

Clearly (AB)' = B'A'

Example # 7 : Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution : Here $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

Let $P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}$

Now $P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, Let $Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

Now $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ -\frac{1}{2} & 0 & -3 \\ -\frac{5}{2} & 3 & 0 \end{bmatrix} = -Q$

Thus $Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$

Thus, B is reresented as the sum of a symmetric and a skew symmetric matrix.

Example # 8 : Show that BAB′ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric (where B is any square matrix whose order is same as that of A).

Solution : Case - I A is symmetric $\Rightarrow$ $A' = A$

$(BAB')' = (B')'A'B' = BAB'$ $\Rightarrow$ BAB′ is symmetric.

Case - II A is skew-symmetric $\Rightarrow$ $A' = -A$

$(BAB')' = (B')'A'B'$

$= B(-A)B'$

$= -(BAB')$

$\Rightarrow$ BAB′ is skew-symmetric

Self practice problems :

(3) For any square matrix A, show that A′A & AA′ are symmetric matrices.

(4) If A & B are symmetric matrices of same order, then show that AB + BA is symmetric and AB – BA is skew-symmetric.

Submatrix : Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

eg. $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$

Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}, \begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}, \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ are all submatrices of A.

Determinant of a square matrix :

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix.

Let $A = [a]_{1\times1}$ be a 1×1 matrix. Determinant A is defined as $|A| = a$.

e.g. $A = [-3]_{1\times1}$ $|A| = -3$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then |A| is defined as ad – bc.

e.g. $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}$, $|A| = 23$

Minors & Cofactors :

Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij}, is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g. 1 $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$M_{11} = d = C_{11}$
$M_{12} = c,\ C_{12} = -c$
$M_{21} = b,\ C_{21} = -b$
$M_{22} = a = C_{22}$

e.g. 2 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}$.

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx) = bx - ay$ etc.

Determinant of any order : Let $A = [a_{ij}]_n$ be a square matrix (n > 1). Determinant of A is defined as the sum of products of elements of any one row (or any one column) with corresponding cofactors.

e.g.1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using first row).

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (using second column).

$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$.

Transpose of a determinant : The transpose of a determinant is the determinant of transpose of the corresponding matrix.

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Properties of determinant :

(1) $|A| = |A'|$ for any square matrix A.

i.e. the value of a determinant remains unaltered, if the rows & columns are inter changed,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$

(2) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

e.g. Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_2 = -D_1$

(3) Let λ be a scalar. Than $\lambda\,|A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by λ

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $E = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $E = KD$

(4) $|AB| = |A|\,|B|$.

(5) $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.

(6) A skew-symmetric matrix of odd order has deteminant value zero.

(7) If a determinant has all the elements zero in any row or column, then its value is zero,

i.e. $D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(8) If a determinant has any two rows (or columns) identical (or proportional), then its value is zero,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(9) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(10) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

i.e. $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. Then $D_2 = D_1$

(11) Let $A = [a_{ij}]_n$. The sum of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly the sum of the products of elements of any column with corresponding cofactors of any other column is zero).

Example # 9 Simplify $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution : Let $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a + b + c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= (a + b + c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$= (a + b + c)\ ((b - c)\ (a - b) - (c - a)^2)$
$= (a + b + c)\ (ab + bc - ca - b^2 - c^2 + 2ca - a^2)$
$= (a + b + c)\ (ab + bc + ca - a^2 - b^2 - c^2) \equiv 3abc - a^3 - b^3 - c^3$

Example # 10 Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Solution : Given detereminant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2,\quad C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a - b)\ (b - c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a - b)\ (b - c)\ [ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 - a^2b - a^2c - ab^2 - abc - b^3 - b^2c]$
$= (a - b)\ (b - c)\ [c(ab + bc + ca) - a(ab + bc + ca)]$
$= (a - b)\ (b - c)\ (c - a)\ (ab + bc + ca)$

Self practice problems

(5) Find the value of $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$.

(6) Simplify $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$.

(7) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)^3$.

(8) Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a - b)(b - c)(c - a)$ by using factor theorem .

Answers : (5) 0 (6) 0

Application of determinants : Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is:

$$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If D = 0 then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines:
$a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Condition for the consistency of three simultaneous linear equations in 2 variables.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Singular & non singular matrix : A square matrix A is said to be singular or non-singular according as |A| is zero or non-zero respectively.

Cofactor matrix & adjoint matrix : Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$
then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} $\forall$ i & j.
Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji}$ $\forall$ i & j.

Properties of cofactor A and adj A:

(a) A . adj A = $|A| I_n$ = (adj A) A where $A = [a_{ij}]_n$.
(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A.
In particular, for 3 × 3 matrix, $|\text{adj } A| = |A|^2$
(c) If A is a symmetric matrix, then adj A are also symmetric matrices.
(d) If A is singular, then adj A is also singular.

Example # 11 : For a 3×3 skew-symmetric matrix A, show that adj A is a symmetric matrix.

Solution : $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ cof $A = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$

adj A = (cof A)′ = $\begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$ which is symmetric.

Inverse of a matrix (reciprocal matrix) :

Let A be a non-singular matrix. Then the matrix $\frac{1}{|A|}$ adj A is the multiplicative inverse of A (we call it inverse of A) and is denoted by A^{-1}.
We have A (adj A) = |A| I_n = (adj A) A

$$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj }A\right) = I_n = \left(\frac{1}{|A|}\text{adj }A\right) A, \text{ for A is non-singular}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \text{ adj } A.$$

Remarks **:**

1. The necessary and sufficient condition for existence of inverse of A is that A is non-singular.

2. A^{-1} is always non-singular.

3. If A = dia $(a_{11}, a_{22}, \ldots, a_{nn})$ where $a_{ii} \neq 0\ \forall$ i, then A^{-1} = diag $(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$.

4. $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also adj (A′) = (adj A)′.

5. $(A^{-1})^{-1} = A$ if A is non-singular.

6. Let k be a non-zero scalar & A be a non-singular matrix. Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.

7. $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.

8. Let A be a non-singular matrix. Then AB = AC ⇒ B = C & BA = CA ⇒ B= C.

9. A is non-singular and symmetric ⇒ A^{-1} is symmetric.

10. $(AB)^{-1} = B^{-1} A^{-1}$ if A and B are non- singular.

11. In general AB = **0** does not imply A = **0** or B = **0**. But if A is non-singular and AB = **0**, then B = **0**. Similarly B is non-singular and AB = **0** ⇒ A = **0**. Therefore, AB = **0** ⇒ either both are singular or one of them is **0**.

Example # 12 : If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A adj A = | A | I. Also find A^{-1}

Solution : We have | A | = 1 (16 – 9) – 3 (4 – 3) + 3 (3 – 4) = 1 ≠ 0
Now $A_{11} = 7, A_{12} = -1, A_{13} = -1, A_{21} = -3, A_{22} = 1, A_{23} = 0, A_{31} = -3, A_{32} = 0, A_{33} = 1$

Therefore $\text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.\ I$

Also $A^{-1}\ \frac{1}{|A|} \text{adj } A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Example # 13 : Show that the matrix A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using the equation, find A^{-1} .

Solution : We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore $AA - 4A = -I$

or $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Example # 14 : For two non-singular matrices A & B, show that adj (AB) = (adj B) (adj A)

Solution : We have (AB) (adj (AB)) = $|AB|\, I_n$

$= |A|\,|B|\, I_n$

$A^{-1}(AB)(\text{adj}(AB)) = |A|\,|B|\,A^{-1}$

$\Rightarrow$ B adj (AB) = |B| adj A $\quad (\because \; A^{-1} = \frac{1}{|A|}\text{ adj } A)$

$\Rightarrow$ $B^{-1}B$ adj (AB) = |B| B^{-1} adj A

$\Rightarrow$ adj (AB) = (adjB) (adj A)

Self practice problems :

(9) If A is non-singular, show that adj (adj A) = $|A|^{n-2} A$.

(10) Prove that adj (A^{-1}) = $(\text{adj } A)^{-1}$.

(11) For any square matrix A, show that |adj (adj A) | = $|A|^{(n-1)^2}$.

(12) If A and B are non-singular matrices, show that $(AB)^{-1} = B^{-1}A^{-1}$.

Elementary row transformation of matrix :

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(c) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

Remarks : Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \approx B$.

Finding inverse using Elementry operations

(i) Using row transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations,

Step I : Write A = IA and

Step II : Apply a sequence of row operation on A = IA till we get, I = BA.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operatdions simultaneously on X and on the first matrix A of the product AB on RHS.

(ii) Using column transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary column operations,

Step I : Write A = AI and

Step II : Apply a sequence of column operations on A = AI till we get, I = AB.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply these row operatdions simultaneously on X and on the second matrix B of the product AB on RHS.

Example # 15 : Obtain the inverse of the matrix A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ using elementary operations.

Solution : Write A = IA, i.e., $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (applying $R_1 \leftrightarrow R_2$)

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 - 3R_1$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_1 \to R_1 - 2R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 + 5R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$ (applying $R_3 \to \frac{1}{2} R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$ (Applying $R_1 \to R_1 + R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$ (Applying $R_2 \to R_2 - 2R_3$)

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

System of linear equations & matrices : Consider the system

$a_{11} x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22} x_2 + ...+ a_{2n} x_n = b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = b_n$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ ..... & ..... & .... & ..... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$.

Then the above system can be expressed in the matrix form as AX = B.
The system is said to be consistent if it has atleast one solution.

System of linear equations and matrix inverse:

If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix.

Results :
(1) If A is non-singular, solution is given by $X = A^{-1}B$.
(2) If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinitely many solutions.
(3) If A is singular and (adj A) B $\neq$ 0, then the system has no solution (we say it is inconsistent).

Homogeneous system and matrix inverse :

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is AX = O. ($\because$ in this case $b_1 = b_2 = ... b_n = 0$), where A is a square matrix.

Results :
(1) If A is non-singular, the system has only the trivial solution (zero solution) X = 0
(2) If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non-trivial solutions.

Example # 16 : Solve the system $\begin{matrix} x + y + z = 6 \\ x - y + z = 2 \\ 2x + y - z = 1 \end{matrix}$ using matrix inverse.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$.

Then the system is AX = B.
|A| = 6. Hence A is non singular.

Cofactor $A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$

adj $A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1} B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ x = 1, y = 2, z = 3.

Self practice problems:

(13) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the inverse of A using |A| and adj A.

(14) Find real values of λ and μ so that the following systems has
(i) unique solution (ii) infinitely many solutions (iii) No solution.

$x + y + z = 6$
$x + 2y + 3z = 1$
$x + 2y + \lambda z = \mu$

(15) Find λ so that the following homogeneous system have a non zero solution

$x + 2y + 3z = \lambda x$
$3x + y + 2z = \lambda y$
$2x + 3y + z = \lambda z$

Answers : (13) $\begin{vmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ -\frac{1}{2} & 3 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{vmatrix}$ (14) (i) $\lambda \neq 3, \mu \in R$ (ii) $\lambda = 3, \mu = 1$ (iii) $\lambda = 3, \mu \neq 1$ (15) $\lambda = 6$

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